Constrained Gain Problem and Its Application to Aircraft Control Systems

S. J. ASSEO*
Cornell Aeronautical Laboratory Inc., Buffalo, N.Y.

The objective of the constrained gain problem is to find optimal gains that satisfy given constraints and minimize a quadratic cost functional of the states. The conditions for optimality are obtained by using the calculus of variations. It is shown that the optimal gains yield a stable closed-loop system and locally minimize the cost functional. An algorithm for the iterative solution of optimal gains is presented and the theory is applied to two aircraft control problems where a comparison is made between present results and those of linear optimal control.

Nomenclature

Vectors

z = n-dimensional state
u = p-dimensional control
b_i = ith column of B
$\delta\zeta,\delta\xi$ = arbitrary variations used in the sufficiency proofs
σ = transformed state
λ = n-dimensional costate

Matrices

A,B = matrices associated with the linear dynamical system
K = matrix of feedback gains
Q,ϕ,R = weighting matrices used in the definition of cost
P = Ricatti matrix
M,N,Ψ,Ω = matrices which appear in the formulation of optimal gains
T = diagonalizing matrix of $(A - BK)$
Λ = diag$(\lambda_1, \ldots, \lambda_n)$ matrix of eigenvalues of $(A - BK)$
S,H = transformation and weighting matrix used in the sufficiency proof

Scalars

I,J = cost functional
$1/\eta,\eta_{ij}$ = iteration stepsize
α = angle of attack
$\dot{\theta}$ = pitch rate
V = airspeed
$\delta_e,\delta_z,\delta_T$ = incremental elevator, flap, and throttle deflection
t_f = interval of integration

Notation

$[\![a_{ij}]\!]$ = matrix A whose i-row j-column element is a_{ij}
δz = variations in z from the extremum
Δz = variations in z between two successive iteration steps
z_j = jth element of vector z
z_m,z_0 = model and initial states
$0[(\Delta z_j)^2]$ = sum of second- and higher-order terms

1. Introduction

IN using optimal control theory to design a control system one must usually satisfy control objectives other than those included in the formulation of a cost functional. For instance, the objective of linear optimal control is to find the control law u which minimizes the quadratic cost functional

$$I = \tfrac{1}{2}\int_0^\infty [z^TQz + u^TRu]dt \tag{1}$$

subject to the differential constraint of the dynamical system $\dot{z} = Az + Bu$, $z(0) = z_0$ where z is the state, A and B are constant matrices, Q and R are arbitrary weighting matrices; Q is positive semidefinite and R is positive definite. The control law derived in Kalman[1] is $u = -Kz$, $K = R^{-1}B^TP$ where P is the positive definite solution of the Riccati equation

$$P(A - BK) + (A - BK)^TP + Q + K^TRK = 0 \tag{2}$$

In addition to minimizing the cost functional in Eq. (1), one may want to 1) provide a specified closed-loop pole pattern, 2) avoid feedback from unmeasurable states, or 3) limit the magnitude of feedback gains. To meet these additional objectives through linear optimal control, one must inevitably resort to a trial-and-error design procedure.

Other methods have been developed to meet these control objectives. In Whitbeck[2] a method is presented for computing feedback gains that yield a specified closed-loop pole pattern. In Kleinman and Athans,[3] a suboptimal control law is obtained that minimizes the cost functional I = Trace (P) where P is the Riccati matrix in Eq. (2). The feedback gain matrix $K(\neq R^{-1}B^TP)$ can be so constructed as to avoid feedback from unmeasurable states. No method has been developed, to the knowledge of the author, to constrain the feedback gains in a specified manner.

This paper is concerned specifically with control applications where a restriction is imposed on the feedback gain magnitude instead of on the magnitude of the control motion. Such control applications can be found 1) in vehicles subjected to small perturbations about a mean trajectory where sufficient control is available to compensate for motion anomalies, 2) in systems with noisy state measurements where one must prevent excessive noise feedback, and 3) in most vehicular control problems where one must avoid high feedback gains in order to prevent excitation of nonlinearities and high-order dynamic modes which are usually neglected in the derivation of a simplified model, etc. Although the methods described in Kalman[1] and Kleinman[3] can be used to satisfy given gain constraints, the constrained gain formulation presented here is more direct.

Following the formulation of the constrained gain problem, an algorithm will be presented for the iterative solution of the optimal gains and the theory will be applied to two aircraft control problems where a comparison will be made between present results and those of linear optimal control.

In the development that follows the symbol $[\![a_{ij}]\!]$ will be used to denote the matrix A whose i-row, j-column element is a_{ij}, superscript T will denote transposition and other symbols will be defined in the text.

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* Principal Engineer, Computer Center Department.

2. Formulation of the Constrained Gain Problem

The objective of the constrained gain problem is to find the optimal feedback gains, k_{ij}, $i = 1, \ldots, p, j = 1, \ldots, n$, which minimize the cost functional

$$J = \frac{1}{2}\int_0^{t_f} z^T Q z dt + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{n} \phi_{ij} k_{ij}^{2\rho} \qquad (3)$$

where Q is positive-definite symmetric matrix ϕ_{ij}'s are positive weights, ρ is a positive integer and z is the n-dimensional state vector. The dynamical system evolves according to the linear state equation

$$\dot{z} = Az + Bu, \qquad z(0) = z_0 \qquad (4)$$

where u is the p-dimensional control and A, B are constant matrices of proper dimensions. A linear feedback control law is assumed

$$u = -Kz \qquad (5)$$

where K is the feedback gain matrix.

The second term in Eq. (3) penalizes high feedback gains. The gain penalty depends on ϕ_{ij} and ρ; for $\rho = 1$ the gain penalty is referred to as soft gain constraints and for $\rho \to \infty$ and $\phi_{ij} = \bar{\phi}_{ij}/(k_{ij\max})2\rho$; where $\bar{\phi}_{ij}$'s are constants and $k_{ij\max}$'s are specified maximum gains, the gain penalty is referred to as hard gain constraints.

The cost functional in Eq. (3) is defined over the infinite time interval, $t_f = \infty$. The results obtained, however, are applicable to the finite time interval case, $t_f < \infty$.

The constrained gain problem is a parameter optimization problem in which the control u is related to the parameter (gain) and to the state z by the control law in Eq. (5). It differs from Pontryagin's[4] parameter optimization problem where the control and the parameter are treated as separate entities. Therefore, the basic principle of calculus of variations[5] will be used in the formulation of the problem.

Necessary Conditions for an Extremum

To proceed with the formulation the constraints in Eqs. (4), (5) are appended to Eq. (3) to obtain the cost function.

$$I = \frac{1}{2}\int_0^{t_f} \{z^T Q z + 2\lambda^T[-\dot{z} + (A - BK)z]\} dt + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{n} \phi_{ij} k_{ij}^{2\rho} \qquad (6)$$

where λ is the n-dimensional costate that enforces the differential constraint in Eq. (4) and the control law in Eq. (5). The variation in cost caused by arbitrary variations in z,λ, and $K(\tilde{z} = z + \delta z, \tilde{\lambda} = \lambda + \delta\lambda, \tilde{K} = K + \delta K)$ from the extremum is written in the usual manner as: $\delta I = \tilde{I}(\tilde{z},\tilde{\lambda},\tilde{K}) - I(z,\lambda,K) = \delta I_1 + \delta I_2$ where

$$\delta I_1 = \frac{1}{2}\int_0^{t_f} \{2\delta z^T[Qz + \dot{\lambda} + (A - BK)^T\lambda] + 2\delta\lambda^T[-\dot{z} + (A - BK)z]\} dt + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{n}\left\{2\rho\phi_{ij}k_{ij}^{2\rho-1} - 2b_i^T\int_0^{t_f}\lambda z_j d\right\} t\delta k_{ij} \qquad (7)$$

δI_1 is the first variation and δI_2 is the second- and higher-order variations of I at the extremum. An expression for the second variation is given below. Equation (7) results from by-parts integration of $\int \lambda^T \delta\dot{z} dt$ with boundary conditions $\lambda(t_f) = 0$; $\delta z(0) = 0$, and by noting that $\int \lambda^T B \delta K z = \Sigma\Sigma b_i^T \int \lambda z_j \delta k_{ij} dt$, where b_i is the ith column of B.

The vanishing of the first variation δI_1 for arbitrary variations δz, $\delta\lambda$, and δK yields the necessary conditions for an extremum[5]:

$$\dot{\lambda} + Qz + (A - BK)^T\lambda = 0 \qquad \lambda(t_f) = 0 \qquad (8a)$$

$$\dot{z} = (A - BK)z \qquad z(0) = z_0 \qquad (8b)$$

$$\rho\phi_{ij}k_{ij}^{2\rho-1} = b_i^T\int_0^{t_f}\lambda z_j dt \qquad (8c)$$

$$i = 1,2, \ldots, p; j = 1,2, \ldots, n$$

where z_j is the jth element of the state vector z.

Equations (8a) and (8b) are the adjoint and state equations which are similar to those associated with a linear optimal control problem. Equation (8c) gives the optimal gains in terms of the state z and costate λ. For hard gain constraints ($\rho = \infty$) the optimal gain expression takes the following form:

$$k_{ij} = \left[\left|\frac{k_{ij\max}{}^{2\rho}}{\rho\bar{\phi}_{ij}}\, b_i^T\int_0^{t_f}\lambda z_j dt\right|\right]^{1/(2\rho-1)} \operatorname{sgn}\left(b_i^T\int_0^{t_f}\lambda z_j dt\right) \qquad (9)$$

where sgn( ) is the sign function defined here as

$$\operatorname{sgn}\begin{bmatrix}+\\-\end{bmatrix} = \begin{bmatrix}+1\\-1\end{bmatrix}, \quad |\operatorname{sgn}(0)| < 1$$

If $b_i^T\int\lambda z_j dt \neq 0$ the quantity in brackets in Eq. (9) tends to $k_{ij\max}$ as $\rho \to \infty$ and the optimal gains tend to $k_{ij} = \pm k_{ij\max}$. If $b_i^T\int\lambda z_j dt = 0$ the optimal gains tend to a value between the bounds $-k_{ij\max} < k_{ij} < +k_{ij\max}$.

The optimal gains in Eq. (8c) depend on the state z and on the costate λ which in turn depend on the optimal gain matrix K through the state equation (8b) and the adjoint Eq. (8a). Hence, the optimal gains must be solved iteratively from the necessary conditions in Eqs. (8). Before presenting an algorithm for the solution of optimal gains, one must show that the optimal gains yield a stable closed-loop system and that the necessary conditions in Eqs. (8) are also sufficient for a local minimum. Stability and sufficiency proofs are given below.

Stability

To prove stability a matrix M is formed from the optimal gains in Eq. (8c).

$$M = \rho[\![\phi_{ij}k_{ij}^{2\rho-1}]\!] = B^T\int_0^{t_f}\lambda z^T dt \qquad (10)$$

The right hand side of Eq. (10) is then evaluated in the appendix as a function of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $(A - BK)$ and the matrix T which diagonalizes $(A - BK)$ as

$$M = B^T T^{T^{-1}}\left[\!\left[\frac{-\omega_{ij}}{\lambda_i + \lambda_j}\right]\!\right]\left[\!\left[\frac{-\psi_{ij}}{\lambda_i + \lambda_j}\right]\!\right] T^T + B^T T^{T^{-1}} N T^T \qquad (11a)$$

where $N = [n_{ij}]$

$$n_{ij} = \frac{1}{\lambda_j - \lambda_i}\sum_{k=1}^{n}\omega_{ij}\psi_{kj}\left(\frac{e^{(\lambda_k+\lambda_j)t_f}}{\lambda_k + \lambda_j} - \frac{e^{(\lambda_k+\lambda_i)t_f}}{\lambda_k + \lambda_i}\right) \qquad (11b)$$

$$\Omega = [\![\omega_{ij}]\!] = T^T Q T \qquad (11c)$$

$$\Psi = [\![\psi_{ij}]\!] = T^{-1}z_0 z_0^T T^{T^{-1}} \qquad (11d)$$

and where z_0 is the initial state and n_{ij},ω_{ij}, and ψ_{ij} are the elements of matrices N,Ω, and Ψ, respectively.

The first matrix in Eq. (11a) is bounded if $\lambda_j + \lambda_i \neq 0$, $i,j = 1, \ldots, n$, regardless of stability. The second matrix, however, increases without bound as $t_f \to \infty$ if $(A - BK)$ contains an unstable root λ_i;

$$\lim_{t_f\to\infty} e^{\lambda_i t_f} \to \infty$$

This violates the restriction imposed on the gain magnitudes. Therefore, if an optimal gain solution exists it must by necessity yield a stable closed-loop system.

Suppose that bounded optimal gains yield an unstable closed-loop root λ_r. It will be shown that this supposition leads to a contradiction, hence, one has a proof of stability. Let real $(\lambda_i + \lambda_r) < 0$, $i = 1, \ldots, n$, $i \neq r$ such that $\lim_{t_f \to \infty} e^{(\lambda_i + \lambda_r)t_f} \to 0$. The elements in the rth row and rth column of N become:

$$n_{rj} = \frac{-\omega_{rr}\psi_{rj}e^{2\lambda_r t_f}}{2\lambda_r(\lambda_j - \lambda_r)} \quad j = 1, \ldots, n;\ j \neq r$$

$$n_{ir} = \frac{\omega_{ir}\psi_{rr}e^{2\lambda_r t_f}}{2\lambda_r(\lambda_r - \lambda_i)} \quad i = 1, \ldots, n;\ i \neq r$$

$$n_{rr} = \frac{-\omega_{rr}\psi_{rr}e^{2\lambda_r t_f}}{4\lambda_r^2}$$

These elements increase without bound unless $\psi_{rj} = 0$, $j = 1, \ldots, n$ while the other elements of N tend to zero as $t_f \to \infty$. (ω_{rr} is always positive since Q and Ω are positive definite.) From Eq. (11d) one has $\psi_{rj} = (u_r^T z_0)(u_j^T z_0)$ where u_j^T is the jth row of T^{-1}, hence in order for the optimal gains to remain bounded despite the unstable root λ_r it is necessary that $u_r^T z_0 = 0$. But, $u_r^T z_0$ is the rth element of the vector $T^{-1}z_0$. Since $e^{\Lambda t}T^{-1}z_0 = T^{-1}e^{(A-BK)t}z_0, u_r^T z_0 = 0$ implies $u_r^T z(t) = 0$. $u_r^T z(t) = 0$ on the other hand implies linear dependence between the state variables $z_1, \ldots, z_n$ which contradicts the definition of an n-dimensional state vector z. Thus, one has a proof of closed-loop stability.

Sufficient Condition for a Local Minimum

To determine the nature of the extremum defined by Eq. (8), one must examine the second variation. Substituting Eq. (8) back into Eq. (7) one gets the following expression for the second variation at the extremum.

$$\delta I_2 = \frac{1}{2}\int_0^{t_f} (\delta z^T Q \delta z - 2\lambda^T B \delta K \delta z)dt + \frac{1}{2}\rho(2\rho - 1)\sum_{i=1}^{p}\sum_{j=1}^{n} \phi_{ij}k_{ij}^{2(\rho-1)}(\delta k_{ij})^2 \quad (12)$$

It will be shown below that the necessary conditions in Eq. (8) are also sufficient for a local minimum when Q is diagonal and $t_f = \infty$. A general sufficiency proof for Q positive definite and $t_f \leq \infty$ is omitted for brevity.

Defining a new set of arbitrary variations $\delta\zeta$ and $\delta\xi$ and using Eq. (8c) to insert the last term in Eq. (12) inside the integral sign, the second variation in Eq. (12) is expressed in the following quadratic form:

$$\delta I_2 = \frac{1}{2}\int_0^{t_f} [\delta\zeta^T \delta\xi^T] H \begin{bmatrix} \delta\zeta \\ \delta\xi \end{bmatrix} dt \quad (13a)$$

where

$$\delta\xi^T = \left[\frac{\delta z_1}{z_1}, \ldots, \frac{\delta z_n}{z_n}\right]$$

$$\delta\zeta^T = \left[\frac{\delta k_{11}}{k_{11}}, \ldots, \frac{\delta k_{p1}}{k_{p1}}, \frac{\delta k_{12}}{k_{12}}, \ldots, \frac{\delta k_{p2}}{k_{p2}}, \ldots, \frac{\delta k_{1n}}{k_{1n}}, \ldots, \frac{\delta k_{pn}}{k_{pn}}\right] \quad (13b)$$

$$H = \left[\begin{array}{c|c} (2\rho - 1)H_{11} & -H_{12} \\ \hline -H_{12}^T & H_{22} \end{array}\right] \quad (13c)$$

Table 1 Feedback gains in example I for $k_{max} = 0.4527$

	k_{11}	k_{12}	k_{21}	k_{22}
Linear optimal control	−0.4527	−0.049	−0.0233	−0.0514
Soft gain constraints	−0.4527	0.339	0.187	−0.327
Hard gain constraints	−0.4527	0.4527	0.4527	−0.4527

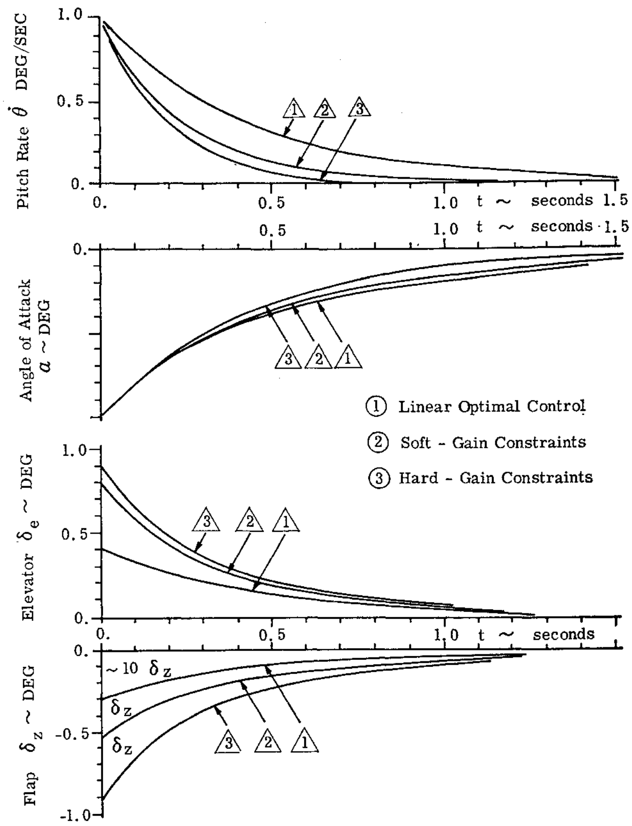

Fig. 1 Transient response—short-period airplane dynamics.

$$H_{11} = \text{diag}[b_1^T\lambda k_{11}z_1, \ldots, b_p^T\lambda k_{p1}z_1, \ldots, b_1^T\lambda k_{1n}z_n, \ldots, b_p^T\lambda k_{pn}]$$

$$H_{22} = \text{diag}[Q_1 z_1^2, \ldots, Q_n z_n^2]$$

$$H_{12} = \begin{bmatrix} b_1^T\lambda k_{11}z_1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ b_p^T\lambda k_{p1}z_1 & 0 & \\ 0 & b_1^T\lambda k_{12}z_2 & \\ \vdots & \vdots & \vdots \\ & b_p^T\lambda k_{p2}z_2 & 0 \\ & 0 & b_1^T\lambda k_{1n}z_n \\ \vdots & \vdots & \vdots \\ 0 & 0 & b_p^T\lambda k_{pn}z_n \end{bmatrix} \quad (13d)$$

and where Q_i's are the elements of Q.

It is desired to prove sufficiency for a local minimum by showing that the second variation in Eq. (13) is strictly positive for arbitrary variations $\delta\zeta$ and $\delta\xi$ (arbitrary δz and δk_{ij}) from the extremum, or equivalently, by showing that all leading principal minors of H are positive.[6] The first $p \times n$ rowed leading principal minors of H are positive since the quadratic form $(2\rho - 1)\delta\zeta^T H_{11}\delta\zeta$ was obtained from the last term in Eq. (12) which is positive for any gain variation δk_{ij}. The remaining leading principal minors of H are given by

$$(2\rho - 1)^{pn-j}\det(H_{11})\prod_{l=1}^{j}[(2\rho - 1)Q_l z_l^2 - \lambda^T B k_l z_l] \quad (14)$$

$$j = 1, \ldots, n$$

where k_l, z_l, and Q_l denote the columns of K, and the elements of z and Q. When $t_f = \infty$, the costate λ is related to the state z by $\lambda = Pz$, where P is a constant matrix which satisfies the algebraic equation[1]

$$P(A - BK) + (A - BK)^TP + Q = 0 \quad (15)$$

From Eqs. (8a), (8b), and (15) one has

$$Q_jz_j^2 - \lambda^TBk_jz_j = -z^T\nu_jz_j$$

where (16)

$$\nu_j = (A - BK)^Tp_j + Pa_j$$

and where p_j, a_j, and k_j are the columns of P, A, and K, respectively.

The positiveness of all leading principal minors of H will be shown by transforming the original states to a different coordinate system. The transformation is $z = S\sigma$ where S is constructed in the following manner: the columns $s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n$ are orthogonal to the vector ν_j defined in Eq. (16), $s_i^T\nu_j = 0$ $i \neq j$, the jth column of S is $s_j = \nu_j$ and the jth row of S is zero except for the jjth element. S is of the form:

$$S = \begin{bmatrix} s_{11} \cdots s_{1,j-1} & \nu_{1j} & s_{1,j+1} \cdots s_{1n} \\ \vdots \quad\quad \vdots & \vdots & \vdots \quad\quad \vdots \\ s_{j-1,1} \cdots s_{j-1,j-1} & \nu_{j-1,j} & s_{j-1,j+1} \cdots s_{j-1,n} \\ 0 \cdots 0 & \nu_{jj} & 0 \cdots 0 \\ s_{j+1,1} \cdots s_{j+1,j-1} & \nu_{j,j+1} & s_{j+1,j+1} \cdots s_{j+1,n} \\ \vdots \quad\quad \vdots & \vdots & \vdots \quad\quad \vdots \\ s_{n1} \cdots s_{n,j-1} & \nu_{jn} & s_{n,j+1} \cdots s_{nn} \end{bmatrix}$$

The transformation S has rank $(S) = n - 1$ because of the way the jth row of S has been prescribed. It is always possible to choose the gain weights ϕ_{ij}, relative to the state weights Q_i to yield sufficiently high feedback gains which satisfy the scalar inequality

$$k_jB^Tp_j > 2a_j^Tp_j, \quad j = 1,2, \ldots, n \quad (17)$$

and thus, insure that $\nu_{jj} = 2a_j^Tp_j - k_j^TB^Tp_j < 0$. Now, substituting $z = S\sigma$ into the quantity in brackets in Eq. (14) and using Eq. (16) one gets:

$$(2\rho - 1)Q_jz_j^2 - \lambda^TBk_jz_j \geq Q_jz_j^2 - \lambda^TBk_jz_j = -\nu_{jj}(\bar{\nu}_j^T\bar{\nu}_j)\sigma_j^2(t) > 0$$

where σ_j, is the jth transformed state. Thus, for sufficiently high gains that satisfy the inequality equation (17) all leading principal minors of H are positive hence, the necessary conditions in Eq. (8) are also sufficient for a local minimum.

This completes the sufficiency proof. An algorithm for the iterative solution of optimal gains is given below. This algorithm computes the proper gain increments Δk_{ij} at each iteration step to insure a monotonically decreasing cost and maintain closed-loop stability at each step of the iterative solution. Thus, closed-loop stability and convergence to a local minimum is achieved through the computation algorithm.

3. Computation Algorithm

A gradient method similar to that described in Kelley[7] is used for the iterative solution of optimal gains from the necessary conditions in Eq. (8). The adjoint and state equations in Eqs. (8a) and (8b) are satisfied and a gain increment ΔK is computed at each iteration step. When Eqs. (8a) and (8b) are satisfied the change in cost between two successive iteration steps is obtained from Eq. (7) by simply replacing δI, δz, and δK by $\Delta I = I^{i+1} - I^i$, $\Delta z = z^{i+1} - z^i$, and $\Delta K = K^{i+1} - K^i$, respectively.

$$I^{i+1} - I^i = \sum_{i=1}^{p}\sum_{j=1}^{n}\left[\rho\phi_{ij}k_{ij}^{2\rho-1} - b_i^T\int_0^{t_f}\lambda z_jdt\right]\Delta k_{ij} + 0[(\Delta z_j)^2,(\Delta k_{ij})^2] \quad (18)$$

where $0(\ldots)$ denotes the sum of second and higher-order terms. The gain increment Δk_{ij} is chosen as

$$\Delta k_{ij} = (1/\eta)[(m_{ij}/\rho\phi_{ij}) - k_{ij}^{2\rho-1}] \quad (19)$$

where m_{ij} are the elements of the matrix M defined in Eq. (10) and (11) and η is a positive constant that determines the iteration stepsize. Now, substituting Eq. (19) back into Eq. (18) yields

$$I^{i+1} - I^i = -\eta\rho\sum_{i=1}^{p}\sum_{j=1}^{n}\phi_{ij}(\Delta k_{ij})^2 + 0[(\Delta z_j)^2,(\Delta k_{ij})^2] \quad (20)$$

For a sufficiently large η (small iteration steps) the first term in Eq. (20) dominates over the higher-order terms and thus, insures a monotonically decreasing cost. For hard gain constraints the gain increment Δk_{ij} takes on a slightly different form

$$\Delta k_{ij} = \eta_{ij}\,\mathrm{sgn}(m_{ij}) \quad (21)$$

where η_{ij} are iteration stepsizes. Equation (21) can be obtained from Eq. (19) by substituting $\eta = |m_{ij}|k_{ij\max}/\rho\bar{\phi}_{ij}\eta_{ij}$ and letting $\rho \to \infty$.

An initial gain matrix K° is selected such that the closed-loop system $A - BK^\circ$ is stable and each gain is smaller in magnitude than the corresponding maximum gains, $|k_{ij}| < k_{ij\max}$. The cost functional, the eigenvalues and the diagonalizing matrix of $A - BK^i$ and the matrix M in Eq. (11) are evaluated at each iteration step by subroutines attributed to Van Ness.[8] The feedback gain matrix is updated in such a way as to maintain closed-loop stability and insure convergence to a local minimum. The computing time on a 360 computer is approximately 2.4 sec/iteration for a sixth-order system; the average convergence time of the program is 1–2 min.

Other computation algorithms were also considered for the solution of the constrained gain problem. The Newton-Raphson method described in Balakrishnan[9] and the second variation method developed in Merriam[10] were not found suitable for the solution of the constrained gain problem.

The constrained gain problem will be applied below to two examples in aircraft control. The first example will demonstrate the advantage of using the constrained gain approach over the linear optimal control approach in obtaining a reduced cost. In the second example the constrained gain problem will be applied to the design of a minimum sensitivity aircraft control system.

4. Applications to Aircraft Control

Example I

Consider the short-period equations of motion of an airplane:

$$\begin{bmatrix}\ddot{\theta}\\ \dot{\alpha}\end{bmatrix} = \begin{bmatrix}-2.0367 & -1.9493\\ 1. & -1.3213\end{bmatrix}\begin{bmatrix}\dot{\theta}\\ \alpha\end{bmatrix} + \begin{bmatrix}-5.3323 & -0.26516\\ -0.16 & -0.2533\end{bmatrix}\begin{bmatrix}\delta_e\\ \delta_z\end{bmatrix}; \quad \begin{matrix}\theta_0 = 1\\ \alpha_0 = -1\end{matrix} \quad (22)$$

where $\dot{\theta}$ is the incremental pitch rate, α is the incremental angle of attack, δ_e is the incremental elevator deflection, and δ_z is the incremental flap deflection, about a trim operating condition. Assume that the control objective here is to find the feedback gains that 1) minimize the cost functional

$$I = \int_0^\infty [\alpha^2 + \dot{\theta}^2]dt \quad (23)$$

and 2) satisfy the gain constraints $|k_{ij}| \leq k_{\max}$, $i,j = 1,2$, where $k_{\max}$ is a specified bound. Both the constrained gain and the linear optimal control methods are used to obtain a solution to the posed problem.

The linear optimal control solution is obtained by solving the Riccati Equation in Eq. (2) several times with various diagonal Q and R matrices until the largest element in the resulting gain matrix equals the specified bound ($k_{11} = k_{\max}$ in this case). A similar procedure is used to obtain the soft gain constraint solution where the matrices Q and ϕ, both diagonal, are adjusted until $k_{11} = k_{\max}$. No adjustment is required to obtain the hard gain constraint solution. The feedback gains thus obtained are listed in Table 1.

The transient response of the system with these feedback gains is plotted in Fig. 1. The maximum gain $k_{\max}$ is changed, each time a new set of feedback gains is computed and the corresponding cost functional in Eq. (23) is plotted in Fig. 2 against $k_{\max}$.

Although the value of the cost functional has limited physical significance it serves the purpose of comparing the relative merits of two alternate designs. For instance, the cost in Eq. (23) is a measure of the system speed of response; the smaller this cost is the faster the system responds to initial conditions.

Examination of Figs. 1 and 2 in conjunction with the gains listed in Table 1 clearly shows that for the same maximum feedback gains derived from diagonal weighting matrices the constrained gain design yields a smaller cost and hence, a faster responding system than the linear optimal control design. The cost reduction here is obtained, of course, at the expense of larger control deflections as noted from Fig. 1.

Example II

The control objective in this example is to force a plant aircraft, whose parameters vary continuously with altitude and airspeed, to follow a model aircraft whose parameters remain approximately fixed. The author has shown[11] that for a given set of plant parameters at some nominal altitude and airspeed the plant aircraft can be made to follow the model aircraft "perfectly" with proper combination of feedback from plant states and feedforward from model states. As the plant aircraft moves away from the nominal operating condition, perfect model following is no longer possible. The objective is then to minimize the error between the motions of plant aircraft and model aircraft in the neighborhood of the nominal operating condition, better known as minimization of plant aircraft motion sensitivity to plant aircraft parameter variations.

The sensitivity vector approach described in Kreindler[12] was ruled out since the control law that results from this approach at least doubles the order of the system. Therefore, both the linear optimal control and the constrained gain design methods were exercised to arrive at a suitable minimum sensitivity design.

The longitudinal linearized equations of motion of the plant and model aircraft are given in first-order form by:

$$\text{plant aircraft} \qquad \dot{x} = Ax + Bu \tag{24a}$$

$$\text{model aircraft} \qquad \dot{x}_m = A_m x_m + B_m u_m \tag{24b}$$

where

$$x = \begin{bmatrix} \dot{\theta} \\ \theta \\ V \\ \alpha \end{bmatrix}; \quad u = \begin{bmatrix} \delta_e \\ \delta_T \\ \delta_z \end{bmatrix}; \quad x_m = \begin{bmatrix} \dot{\theta}_m \\ \theta_m \\ V_m \\ \alpha_m \end{bmatrix}; \quad u_m = \begin{bmatrix} \delta_{em} \\ \delta_{Tm} \end{bmatrix} \tag{24c}$$

The vectors x, x_m, u, and u_m in this equation denote increments in motion and control variables from trim condition; θ is pitch $\dot{\theta}$ is pitch rate, V is airspeed, α is angle of attack, δ_e is elevator control, δ_T is throttle control and δ_z is flap control.

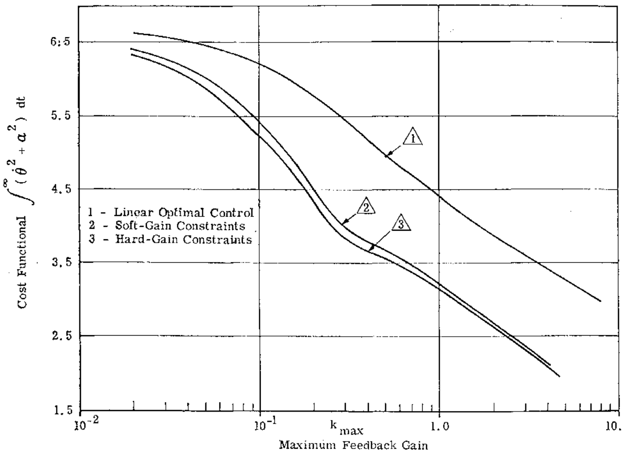

Fig. 2 Cost vs maximum feedback gain.

(A, B, A_m, B_m are constant matrices of appropriate dimensions which contain aircraft stability derivatives computed at a nominal operating condition.) Note that the plant aircraft uses the flaps as an active controller in addition to elevator and throttle to provide linearly independent controls for each degree of freedom. It was shown[11] that proper combination of feedback from $\int\theta$, $\int V$, $\int\alpha$ and feedforward from $\int\theta_m$, $\int V_m$, $\int\alpha_m$ reduces motion sensitivity of the plant aircraft. These variables are appended to the motion variables in Eq. (24c) to obtain the total dynamical description of the plant aircraft.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

where

$$y = \begin{bmatrix} \int\theta dt \\ \int V dt \\ \int \alpha dt \end{bmatrix}; \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{25}$$

The feedback portion of the control law is

$$u = -K \begin{bmatrix} x \\ y \end{bmatrix} \tag{26}$$

where the feedback gains k_{ij}, $i = 1,2,3$, $j = 1, \ldots, 7$ are obtained by minimizing the cost functionals:

$$\text{Linear optimal control:} \quad I = \frac{1}{2} \int_0^\infty (x^T Q x + u^T R u) dt$$

$$\text{Constrained gain:} \quad I = \frac{1}{2} \int_0^\infty x^T Q x dt + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{7} \phi_{ij} k_{ij}^{2p} \tag{27}$$

The constraints on the gain magnitudes were selected apriori. The linear optimal control solution was obtained by solving the Riccati Equation in Eq. (2) several times with various diagonal Q's and R's until the constraints on the gain magnitudes were met. Several solutions were obtained using the constrained gain approach by adjusting the diagonal weighting matrices (Q and ϕ for soft constraints, Q for hard constraints) and the initial conditions. The best designs in each case were chosen for comparison. The resulting feedback matrices are listed in Table 2.

The parameters in A and B in Eq. (24a) were then varied from the nominal to examine the sensitivity of plant aircraft motion to these parameter variations. The aircraft transient responses to a step throttle input are plotted in Figs. 3, 4, and 5. Figure 3 shows the time history of plant aircraft and model aircraft motions from which the quality of model following is evident. Figures 4 and 5 show the error in air-

Table 2 Feedback gain matrices used in example II

	Feedback gain matrices						
K = linear opt. cont.	−2.556	−10.34	2.494	−3.186	0	0.6472	−4.01
	−0.1875	−5.914	16.08	4.58	0	9.605	2.746
	−0.0726	4.624	2.523	−8.114	0	2.703	−8.747
K (soft gain) = curve 1 Fig. 5	−2.542	−10.284	0	0	0	0	0
	0	0	16.034	0	0	5.83	0
	0	0	0.0192	−8.07	0	0	−8.7
K (hard gain) = curve 2 Fig. 5	−3.	−11.35	0	0	0	0	−2.032
	0	0	20.	10.	0	20.	10.06
	0	0	0	−10.	0	0	−20.
K (hard gain) = curve 3 Fig. 5	−2.406	−15.	0	0	−20.	0	0
	0	0	20.	0	0	25.	0
	−3.	0	0	−10.	0	0	−50

craft motion variables caused by parameter changes; the gains in Fig. 4 were computed using linear optimal control and the gains in Fig. 5 were obtained using the constrained gain approach. Examination of these figures clearly shows that 1) although the maximum gains in any given column of K are of comparable magnitude, the design obtained through soft gain constraints is considerably less sensitive than the linear optimal control design, 2) further sensitivity reduction is obtained through hard gain constraints by allowing a slight increase in gain magnitudes and especially by using $\int\theta(\tau)d\tau$ feedback.

5. Conclusions and Discussion

The results obtained in this paper are summarized as follows: 1) The constrained gain problem is formulated, necessary and sufficient conditions for a local minimum are obtained and it is shown that the optimal gains derived from the necessary conditions yield a stable closed-loop system. 2) A rapidly converging algorithm is developed. The gain increments and the cost functional are evaluated in closed form as a function of the eigenvalues and the modal matrix of $(A - BK)$; no numerical integration is required in the iterative solution. 3) The constrained gain problem is applied to two aircraft control problems where a comparison is made between present results and those of linear optimal control. It is shown that when one of the control objectives is to restrict the gain magnitude the constrained gain approach yields a smaller cost and a faster responding system. It is further shown that the latter method is more effective than linear optimal control in reducing motion sensitivity to plant parameter variations.

One drawback of the constrained gain method is the dependence of optimal gains on the initial states. However, initial states are free parameters at the designer's disposal, which can be so adjusted as to obtain the best possible design.

Parameter adjustment is usually required in both constrained gain and linear optimal control to obtain satisfactory system performance with acceptable gains; in linear optimal control the weighting matrices Q and R must be selected, and in constrained gain the matrices Q and ϕ and the initial conditions must be determined. If maximum gains are specified the constrained gain approach requires the selection of fewer parameters for comparable system performance. For instance, in example II diagonal weighting matrices were used in both methods, the constrained gain method yielded a less sensitive system. To obtain a comparable system through linear optimal control one must use nondiagonal Q and R matrices or allow state-control quadratic cross product terms of the form $2x^TSu$ in the cost functional. The number of free parameters, in this case, compares as follows: constrained gain problem requires $2n$ free parameters (n diagonal elements of Q and n initial states), linear optimal control derived from $\int(x^TQx + u^TRu)dt$ requires $n(n+1)/2 + p(p+1)/2$ and optimal control derived from $\int(x^TQx +$

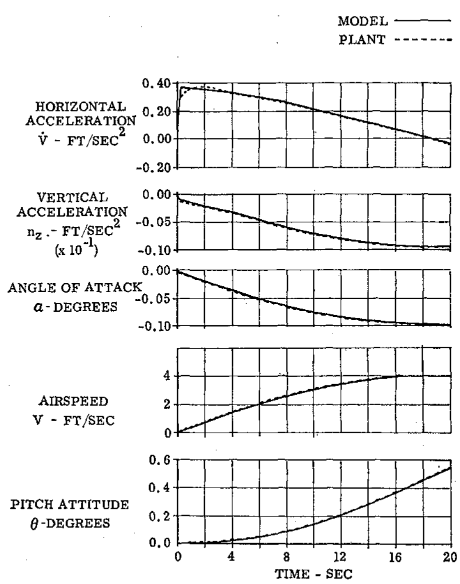

Fig. 3 Aircraft throttle step response.

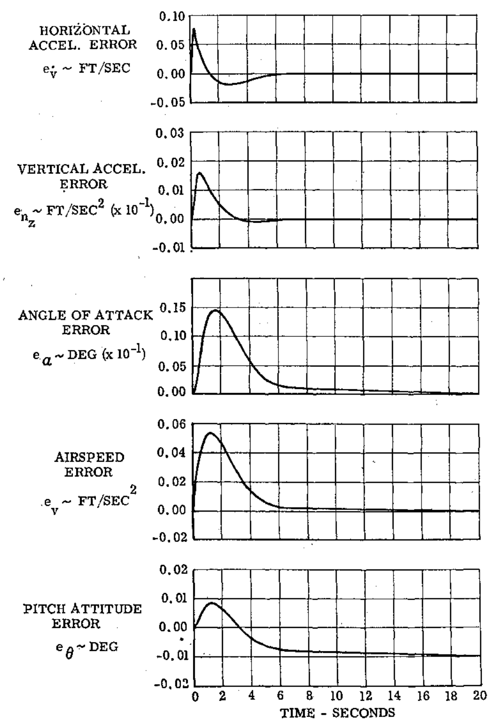

Fig. 4 Error time history, throttle step input, linear optimal control.

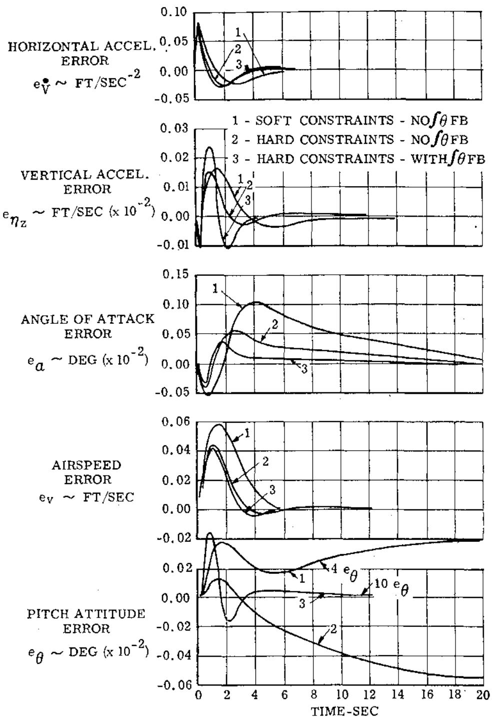

Fig. 5 Error time history, throttle step input, otpimal gain.

$2x^TSu + u^TRu)dt$ requires $n(n+1)/2 + p(p+1)/2 + pn$ free parameters. In example II the number of free parameters are: 8 for constrained gain, 16 for linear optimal control with $S = 0$ and 28 for linear optimal control with $S \neq 0$. Thus, the constrained gain method is more direct in obtaining a solution to control problems where restrictions are imposed on feedback gains.

One may argue that maximum gains are additional free parameters to be selected by the designer, thus, increasing the total number of free parameters required for the constrained gain problem. A similar situation exists, however, in linear optimal control where the designer is forced to pick the best solution on the basis of good performance and acceptable gains. The "acceptable gain" criterion in this case is not as well defined as in the constrained gain case.

The constrained gain problem has multiple solutions; the particular solution obtained depends on the initial gain matrix K° used in the iterative solution. Experience has shown, however, that, predominant gains usually converge to nearly the same values while the less significant gains converge to one of several local minima.

The constrained gain method can be easily applied to the design of incomplete-state feedback systems. To avoid feedback from unmeasurable states, one simply sets the corresponding maximum gains to zero, for instance, if x_3 is unmeasurable one sets $k_{i3\max} = 0$, $i = 1, \ldots, p$, and one uses the computation algorithm developed to obtain a control system with no feedback from x_3.

Appendix

Closed form expressions for the cost functional and the optimal gains are derived below. An expression for the cost functional is given in Kalman[1]

$$I = \frac{1}{2}\int_0^\infty z^TQzdt = \frac{1}{2} z_0{}^TPz_0 \qquad \text{(A1)}$$

where P is a constant matrix which satisfies Eq. (15).

A matrix M is constructed from the optimal gains in Eq. (8c)

$$M = \rho[\![\phi_{ij}k_{ij}{}^{2\rho-1}]\!] = B^T\int_0^{t_f} \lambda z^T dt \qquad \text{(A2)}$$

The solution of the adjoint equation [Eq.(8a)] is given by

$$\lambda(t) = \int_t^{t_f} e^{-(A-BK)^T(t-\tau)}Qe^{-(A-BK)(t-\tau)}dt \cdot z(t) \qquad \text{(A3)}$$

$\lambda(t)$ is evaluated in terms of the matrix of eigenvalues $\Lambda = \text{diag}[\lambda_1, \ldots, \lambda_n]$ and the matrix T which diagonalizes $(A - BK)$ as follows:

$$\lambda(t) = T^{T^{-1}}\left[\!\left[\frac{e^{(\lambda_i+\lambda_j)(t_f-t)} - 1}{\lambda_i + \lambda_j}\,\omega_{ij}\right]\!\right]T^{-1}z(t) \qquad \text{(A4)}$$

where

$$\Omega = T^TQT = [\![\omega_{ij}]\!]$$

similarly, $z(t)$ is obtained from the state equation (8b)

$$z(t) = Te^{\Lambda t}T^{-1}z_0 \qquad \text{(A5)}$$

Now, substituting Eqs. (A4) and (A5) into Eq. (A2) and integrating yields the desired closed form expression for M.

$$M = B^TT^{T^{-1}}\left[\!\left[\frac{-\omega_{ij}}{\lambda_i+\lambda_j}\right]\!\right]\left[\!\left[\frac{-\psi_{ij}}{\lambda_i+\lambda_j}\right]\!\right]T^T +$$
$$B^TT^{T^{-1}}\left[\!\left[\frac{1}{\lambda_j-\lambda_i}\sum_{k=1}^{n}\omega_{ik}\psi_{kj}\left(\frac{e^{(\lambda_k+\lambda_j)t_f}}{\lambda_k+\lambda_j} - \frac{e^{(\lambda_k+\lambda_i)t_f}}{\lambda_k+\lambda_i}\right)\right]\!\right]T^T \qquad \text{(A6)}$$

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